Logarithms and Exponents [256 marks]

1. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3\sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve
$\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

2. [Maximum mark: 5]

Solve the equation $2\ln x = \ln 9 + 4$. Give your answer in the form $x = pe^q$ where $p, q \in \mathbb{Z}^+$. [5]

3. [Maximum mark: 6]

(a) Given that $x > 7$, show that $\frac{x}{x^2-8x+7} \times \frac{x^2-1}{x+1} \equiv \frac{x}{x-7}$. [2]

(b) Hence, or otherwise, solve
$\log_2\left[x\left(x^2-1\right)\right] - 1 = \log_2\left[\left(x^2-8x+7\right)(x+1)\right]$
. [4]

4. [Maximum mark: 5]

Solve the equation $3\log_8 10x - \log_4 x = 1$ for $x > 0$. [5]

5. [Maximum mark: 14]

A population of frogs, F, in a swamp after t months, can be modelled by the function

$F(t) = 1850 \times 1.105^t$ where $t \geq 0$.

(a) Find the population of frogs after one year. [2]

(b) After x complete months, the population will be at least $35\,000$ frogs. Find the value of x. [3]

The function F can be written in the form $F(t) = 1850\mathrm{e}^{kt}$.

(c) Find the exact value of k. [2]

(d) Find the rate at which the population of frogs is growing after 15 months. [2]

A more realistic model describing the population of frogs, G, after t months is given by

$G(t) = \frac{35\,000}{1+A\mathrm{e}^{-0.0998t}}$ where $t \geq 0$.

(e) After 15 months, this model predicts a population of 6995 frogs. Find the value of A.

[2]

(f) Find the value of t when the rate of population growth is the greatest. [2]

(g) By considering the graph of G or otherwise, state one reason why $G(t)$ is a more appropriate long-term model than $F(t)$.

[1]

6. [Maximum mark: 5]
Write each of the following expressions in the form $\ln\ k$, where $k \in \mathbb{Z}^{+}$.

(a) $\ln 3 + \ln 4$ [1]

(b) $3 \ln 2$ [2]

(c) $-\ln \frac{1}{2}$ [2]

7. [Maximum mark: 15]

The function f is defined by $f(x) = 5(x+1)(x+3)$, where $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a\left(x - h^2\right) + k$, where $a, h, k \in \mathbb{Z}$. [4]

(b) Sketch the graph of $y = f(x)$, showing the values of any intercepts with the axes and the coordinates of the vertex. [4]

(c) Solve the inequality $f(x) \leq 40$. [4]

The function g is defined by $g(x) = \ln x$, where $x \in \mathbb{R}, x > 0$.

(d.i) Write down an expression for $(f \circ g)(x)$. [1]

(d.ii) Solve the inequality $(f \circ g)(x) \leq 40$. [2]

8. [Maximum mark: 17]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$.

(a) Find $f^{-1}(8)$. Express your answer in the form $\frac{p}{q}$ where $p,\ q \in \mathbb{Z}$. [3]

The function g is defined by $g(x) = 1 + \log_2 x$, where $x \in \mathbb{R}^+$.

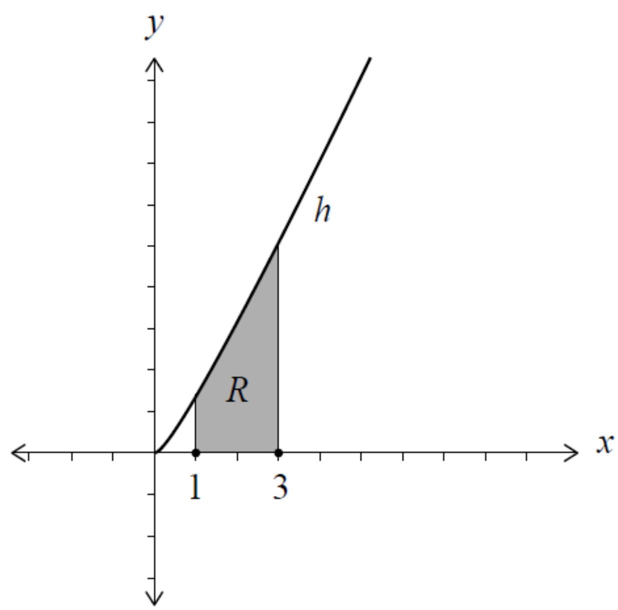
(b.i) Find an expression for $g^{-1}(x)$. [2]

(b.ii) Describe a sequence of transformations that transforms the graph of $y = g^{-1}(x)$ to the graph of $y = f(x)$. [2]

(c) Show that $(f \circ g)(x) = 4x^2$. [3]

The function h is defined by $h(x) = \frac{4x^2}{2x+1},\ x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of h. Let R be the region enclosed by the graph of h and the x-axis, between the lines $x = 1$ and $x = 3$.



(d.i) Show that $2x - 1 + \frac{1}{2x+1} = \frac{4x^2}{2x+1}$. [2]

(d.ii) Hence or otherwise, find the area of R, giving your answer in the form $p + q \ln r$, where $p,\ q,\ r \in \mathbb{Q}^+$. [5]

9. [Maximum mark: 5]

Let $\log_{10} 2 = p$ and $\log_{10} 3 = q$.

(a) Find an expression for $\log_{10} 24$ in terms of p and q. [3]

(b) Find an expression for $\log_3 8$ in terms of p and q. [2]

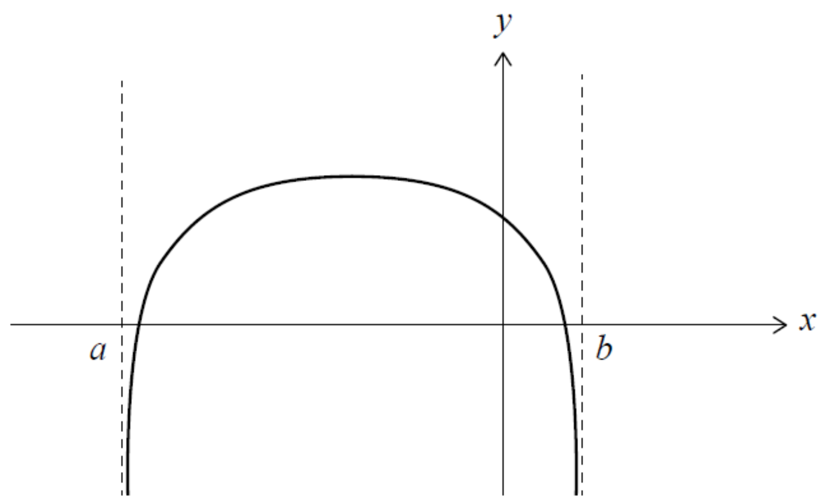
10. [Maximum mark: 16]

(a.i) Solve $5 - 4x - x^2 = 0$. [2]

(a.ii) Hence, find the values of x such that $5 - 4x - x^2 > 0$. [2]

Consider the function $f(x) = \log_k (5 - 4x - x^2)$, where $a < x < b$ and $k > 1$.

Part of the graph of f is shown in the following diagram.



The graph of f has vertical asymptotes at $x = a$ and $x = b$.

(b) Write down the value of

(b.i) a; [1]

(b.ii) b. [1]

(c) Find the exact values of x such that $f(x) = 0$. [4]

The graph of f has a maximum value of 2.

(d) Find the value of k. [6]

11. [Maximum mark: 14]

The function f is defined as $f(x) = \log_2 (4x)$, where $x > 0$.

(a) Find the value of

(a.i) $f(8)$; [2]

(a.ii) $f\left(\frac{1}{4}\right)$. [1]

(b) Find an expression for $f^{-1}(x)$. [4]

(c) Hence, or otherwise, find $f^{-1}(0)$. [1]

The graph of $y = f\left(16x^3\right)$ can be obtained by translating and stretching the graph of $y = \log_2 x$.

(d) Describe these two transformations specifying the order in which they are to be applied. [6]

12. [Maximum mark: 5]

Consider the function $h(x) = \log_{10} \left(3x^2 - rx + r - 2\right)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

13. [Maximum mark: 5]

Consider the function $h(x) = \log_{10} \left(4x^2 - rx + r - 1\right)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

14. [Maximum mark: 5]

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10} \left(\frac{1}{a}\right)$; [2]

(b) $\log_{1000} a$. [3]

15. [Maximum mark: 16]

Consider the arithmetic sequence $a,\ p,\ q \ldots,$ where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Consider the geometric sequence $a,\ s,\ t \ldots,$ where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Consider the case where $a = 9,\ s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(d.i) arithmetic sequence; [2]

(d.ii) geometric sequence. [2]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e.i) Find the common difference of the new sequence in terms of $\ln 3$. [3]

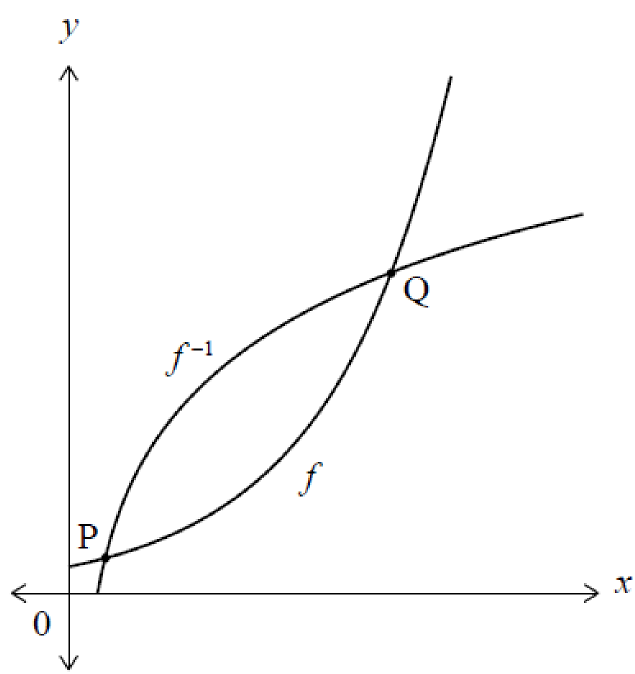
(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3$. [3]

16. [Maximum mark: 12]

Consider the function defined by $f(x) = \frac{3}{2}e^{x-2},\ 0 \le x \le 4$.

(a) Show that the inverse function is given by $f^{-1}(x) = 2 + \ln\left(\frac{2x}{3}\right)$. [3]

The graphs of f and f^{-1} intersect at two points P and Q, as shown on the following diagram.



(b) Find PQ. [3]

The graph of f is reflected in the x-axis and then translated parallel to the y-axis by 5 units in the positive direction to give the graph of a function g.

(c) Write down

(c.i) an expression for $g(x)$; [2]

(c.ii) the domain of g. [1]

(d) Solve the equation $f(x) = g(x)$. Give your answer in the form $x = a + \ln b$, where $a,\ b\ \in \mathbb{Q}$. [3]

17. [Maximum mark: 6]

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10\log_{10}\left(I \times 10^{12}\right)$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

(b) Determine the loudness of S_2. [2]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

18. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 7)$, where $x > \frac{7}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0,\ d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

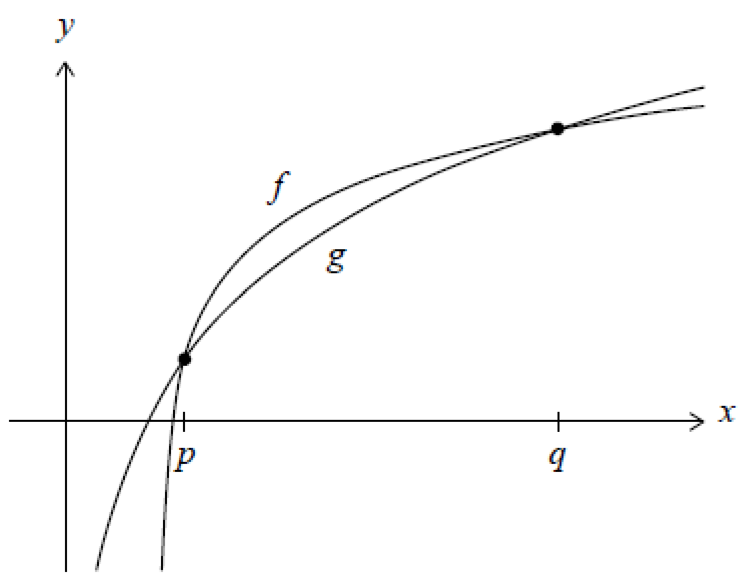
The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b.i) Show that, at the points of intersection, $x^2 - 2dx + 7d = 0$. [4]

(b.ii) Hence, show that $d^2 - 7d > 0$. [3]

(b.iii) Find the range of possible values of d. [2]

The following diagram shows parts of the graph $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

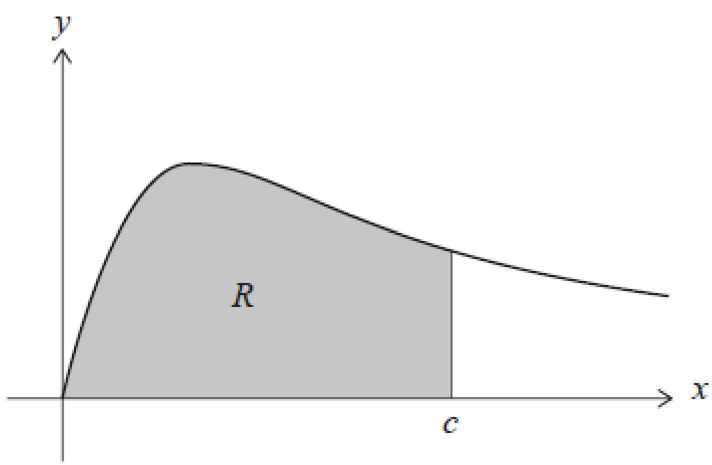
(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a,\ b \in \mathbb{Z}^+$. [5]

19. [Maximum mark: 6]

Find the range of possible values of k such that $e^{2x} + \ln k = 3e^{x}$ has at least one real solution. [6]

20. [Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2+2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c. [6]

21. [Maximum mark: 15]

Calculate the value of each of the following logarithms:

(a.i) $\log_2 \frac{1}{16}$. [2]

(a.ii) $\log_9 3$. [2]

(a.iii) $\log_{\sqrt{3}} 81$. [3]

It is given that $\log_{ab} a = 3$, where $a,\ b \in \mathbb{R}^+,\ ab \neq 1$.

(b.i) Show that $\log_{ab} b = -2$. [4]

(b.ii) Hence find the value of $\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}$. [4]

22. [Maximum mark: 5]

(a) The expression $\frac{3\sqrt{x}-5}{\sqrt{x}}$ can be written as $3 - 5x^p$. Write down the value of p. [1]

(b) Hence, find the value of $\int_1^9 \left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right) \mathrm{d}\ x$. [4]

23. [Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)\mathrm{e}^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5\mathrm{e}^k x$.

Find the value of k. [5]

24. [Maximum mark: 15]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm\frac{1}{\sqrt{3}}$. [2]

(a.ii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

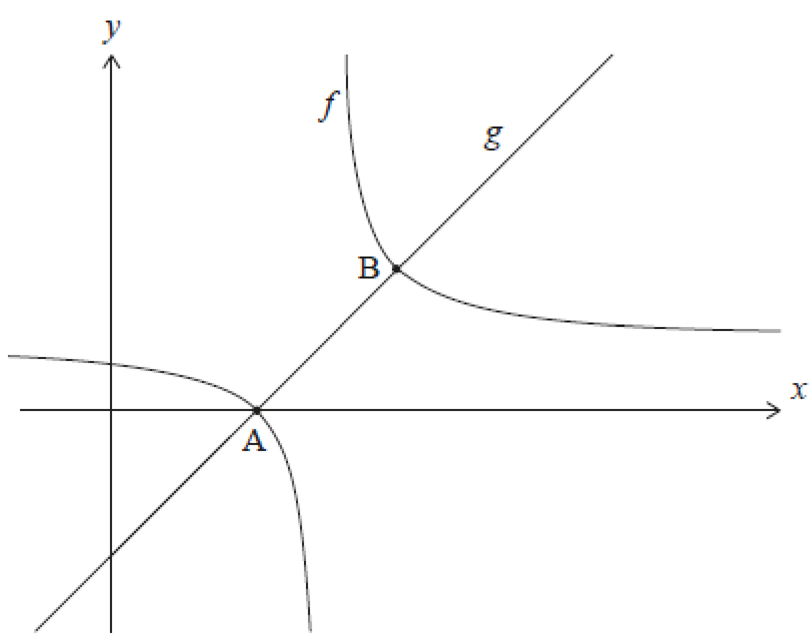
(b.iii) The sum of the first n terms of the series is $-3 \ln x$.

Find the value of n. [6]

25. [Maximum mark: 15]

Consider the functions $f(x) = \frac{1}{x-4} + 1$, for $x \neq 4$, and $g(x) = x - 3$ for $x \in \mathbb{R}$.

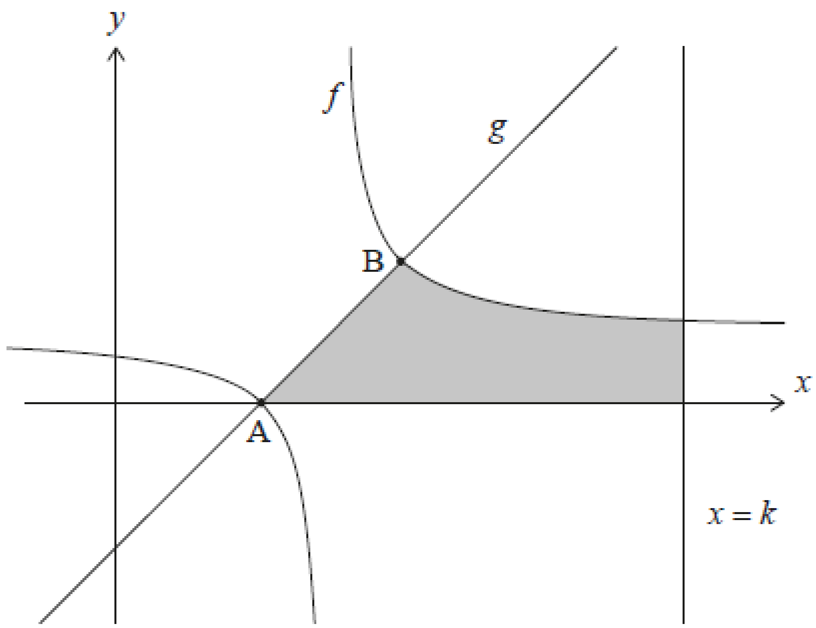
The following diagram shows the graphs of f and g.



The graphs of f and g intersect at points A and B. The coordinates of A are $(3,\ 0)$.

(a) Find the coordinates of B. [5]

In the following diagram, the shaded region is enclosed by the graph of f, the graph of g, the x-axis, and the line $x = k$, where $k \in \mathbb{Z}$.



The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

(b) Find the value of k and the value of p. [10]

26. [Maximum mark: 15]

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3},\ 4\right)$.

(a) Show that $a = 8$. [2]

(b) Write down an expression for $f^{-1}(x)$. [1]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Consider the arithmetic sequence
$\log_8 27\ ,\ \ \log_8 p\ ,\ \ \log_8 q\ ,\ \ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

(d.ii) Find the value of p and the value of q. [5]